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A study on Some Difference sequence spaces and their Matrix transformations over Non –Archimedean fields

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Abstract— In this paper we define some difference sequence spaces over non-Archimedean fields and the matrix transformations of these sequence spaces, where the sequences, series and infinite matrices have entries in a complete non-trivially valued non-Archimedean field K as in [7].

Index Terms— Bounded,convergent and null sequence spaces, Difference sequence spaces, Matrix transformations, Non-trivially valued non-Archimedean field.

1 Introduction

Let l_∞, c and c_0 denote the Banach spaces of bounded convergent and null squenes $x = (x_k) \in K, k = 1,2...,$ with $\|x\| = \sup_k |x_k|$. In this paper we extend the difference sequence spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ of [5] to $\Delta l_\infty(p), \Delta c(p)$ and $\Delta c_o(p)$, where $p = (p_k)$ be a bounded sequence of strictly positive real numbers .i.e., $1 \le p_k \le \sup p_k < \infty$.

If $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, then these spaces are defined as follows:

$$\Delta l_\infty(p) = \left\{ x = (x_k) : \sup_k |\Delta x_k|^{p_k} < \infty \right\}$$

$$\Delta c(p) = \{ x = (x_k) : |\Delta x_k - \alpha_k|^{p_k} \to 0 \text{ as } k \to \infty \text{ for some } \alpha_k \in K \}$$

and $\Delta c_0(p) = \left\{ x = (x_k) : |\Delta x_k|^{p_k} \to 0 \text{ as } k \to \infty \right\}$, when $\sup_k p_k < \infty$.

When $p_k = c$, a constant, k=1,2,…,

$\Delta l_\infty(p) = l_\infty(\Delta), \Delta c(p) = c(\Delta)$ and $\Delta c_0(p) = c_0(\Delta)$

and the above difference sequence spaces are non-Archimedean Banach spaces with the norm $\|x\|_\Delta = |x_1| + \|\Delta x\|$.

Here we prove that $(\Delta l_\infty(p), \|.\|_\Delta)$ is a non-Archimedean Banach space.

Let (x^n) be a Cauchy sequence in $\Delta l_\infty(p)$, where $x^n = (x_i^n) = (x_1^n, x_2^n, ...) \in \Delta l_\infty(p)$.

Then for each $n \in \mathbb{N}$, we have

$$\|x^n - x^m\|_\Delta = |x_1^n - x_1^m|^{p_k} + \|\Delta x^n - \Delta x^m\|_\infty \to 0 \text{ as } n, m \to \infty.$$

This implies that $|x_1^n - x_1^m|^{p_k} \to 0$ as $n, m \to \infty$, and hence $|x_k^n - x_k^m|^{p_k} \to 0$ as $n, m \to \infty$, then by the completeness of K, (x_k^n) converges to some x_k in K.

i.e., $\lim_{n\to\infty} x_k^n = x_k$, k =1,2….

Further for each $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ such that for all $n,m \ge N$ and for all $k \in \mathbb{N}$.

$$|x_1^n - x_1^m|^{p_k} < \varepsilon, \quad |x_{k+1}^n - x_{k+1}^m - (x_k^n - x_k^m)|^{p_k} < \varepsilon$$

Then as $m \to \infty$, we have,

$$\lim_m |x_1^n - x_1^m|^{p_k} = |x_1^n - x_1|^{p_k} \le \varepsilon \quad (1)$$

and

$$\lim_m |x_{k+1}^n - x_{k+1}^m - (x_k^n - x_k^m)|^{p_k} = |x_{k+1}^n - x_{k+1} - (x_k^n - x_k|^{p_k} \le \varepsilon,$$

for all $n \ge N$. (2)

since ε is not dependent on k, We have

$$\sup_k |x_{k+1}^n - x_{k+1} - (x_k^n - x_k)|^{p_k} \le \varepsilon$$

$$\Rightarrow \|\Delta x^n - \Delta x\|_\infty \le \varepsilon \quad (3)$$

Consequently we have ,

$$\|x^n - x\|_\Delta = |x_1^n - x_1|^{p_k} + \|\Delta x^n - \Delta x\|_\infty \le 2\varepsilon, \quad \text{using(1)\&(3).}$$

Hence we obtain $x^n \to x (n \to \infty)$ in $\Delta l_\infty(p)$, where $x = (x_k)$.

Now we must show that $x \in \Delta l_\infty(p)$. For, we consider

$$|x_k - x_{k+1}|^{p_k} = |x_k - x_k^N + x_k^N - x_{k+1}^N + x_{k+1}^N - x_{k+1}|^{p_k}$$

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$$\leq \max\left\{\left|x_k{}^N - x_{k+1}{}^N\right|^{p_k}, \left|x_{k+1}{}^N - x_k{}^N - (x_{k+1} - x_k)\right|^{p_k}\right\}$$

$$\leq \sup_k \left\{\left|x_k{}^N - x_{k+1}{}^N\right|^{p_k}, \left|x_{k+1}{}^N - x_k{}^N - (x_{k+1} - x_k)\right|^{p_k}\right\}$$

$\leq \varepsilon$, for all $k \in \mathbb{N}$

That is., $\sup_k |x_k - x_{k+1}|^{p_k} < \infty$.

Hence $\Delta x_k \in l_\infty(p)$ implies that $x \in \Delta l_\infty(p)$.

1.1.Lemma:

$\sup_k |x_k - x_{k+1}|^{p_k} < \infty$ iff

(i) $\sup_k k^{-1}|x_k|^{p_k} < \infty$ and

(ii) $\sup_k |x_k - k(k+1)^{-1}x_{k+1}|^{p_k} < \infty$.

Proof:

Let $\sup_k |x_k - x_{k+1}|^{p_k} < \infty$.

Then for each $k \in \mathbb{N}$ there exists an integer $M > 0$ such that $|x_k - x_{k+1}|^{p_k} \leq kM$. (4)

To prove (i):

Now,

$$|x_1 - x_{k+1}|^{p_k} = \left|\sum_{\nu=1}^{k}(x_\nu - x_{\nu+1})\right|^{p_k}$$

$$\leq \max\left\{ |x_1 - x_2|^{p_k}, |x_2 - x_3|^{p_k}, \ldots\ldots |x_k - x_{k+1}|^{p_k} \right\}$$

$$\leq \max\{ M, 2M, \ldots\ldots\ldots, kM \}, \quad \text{using(4)}$$

$$\leq kM$$

$$\Rightarrow k^{-1}|x_1 - x_{k+1}|^{p_k} \leq M$$

$$\Rightarrow \sup_k k^{-1}|x_k|^{p_k} < \infty.$$

To prove (ii):

Now for each $k \in \mathbb{N}$ we choose an integer M such that

$$M > \max(1, \sup_k k^{-1}|x_k|^{p_k}). \quad (5)$$

Then we have

$$\left|x_k - k(k+1)^{-1}x_{k+1}\right|^{p_k}$$

$$= \left|k(k+1)^{-1}(x_k - x_{k+1}) + (k+1)^{-1}x_k\right|^{p_k}$$

$$= (k+1)^{-1}\left|k(x_k - x_{k+1}) + x_k\right|^{p_k}$$

$$\leq (k+1)^{-1} \max\left\{ k|x_k - x_{k+1}|^{p_k}, |x_k|^{p_k} \right\}$$

$$\leq k(k+1)^{-1} \max\left\{ |\Delta x_k|^{p_k}, k^{-1}|x_k|^{p_k} \right\}$$

$$\leq k(k+1)^{-1}\max(kM, M), \quad \text{using(4)\&(5)}$$

$$\leq \frac{k^2}{(k+1)}M$$

i.e., $\sup_k \left|x_k - k(k+1)^{-1}x_{k+1}\right|^{p_k} < \infty$

Conversely suppose that (i) and (ii) hold, then there exists integers M_1 (>0) and M_2 (>0) such that

$$k^{-1}|x_k|^{p_k} < M_1 \text{ and } \left|x_k - k(k+1)^{-1}x_{k+1}\right|^{p_k} < M_2 \quad (6)$$

Now,we have,

$$|x_k - x_{k+1}|^{p_k} = \frac{k+1}{k}\left|k(k+1)^{-1}x_k - k(k+1)^{-1}x_{k+1}\right|^{p_k}$$

$$= \frac{k+1}{k}\left|x_k - (k+1)^{-1}x_k - k(k+1)^{-1}x_{k+1}\right|^{p_k}$$

$$= \frac{k+1}{k}\max\left\{\left|x_k - k(k+1)^{-1}x_{k+1}\right|^{p_k}, (k+1)^{-1}|x_k|^{p_k}\right\}$$

$$\leq \frac{k+1}{k}\max\{M1, M2\}, \quad \text{using(6)}$$

$$\leq \frac{k+1}{k}M \quad \text{where, } M = \max(M1, M2)$$

$$\Rightarrow Sup|x_k - x_{k+1}|^{p_k} < \infty.$$

Which completes the proof of the lemma.

2 Matrix maps:

If X and Y are any two sequence spaces, then(X, Y) denote the class of all infinite matrices $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 1,2\ldots$, that maps X in toY, i.e., for which the series $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$ coverges for all $x \in X$ and all $n = 1,2\ldots$, and such that $Ax = (Ax)_n \in Y$ for all $x \in X$ .Here Ax is called the matrix transformation of the sequence $x = (x_k)$.

2.1.Theorem:

Let $p_k > 0$ for every k = 1,2,…, then $A \in (\Delta l_\infty(p), l_\infty)$ iff for every integer N>1,

$$\sup_k k|a_{nk}|N^{1/p_k} < \infty . \qquad (7)$$

Proof:

Suppose that (7) holds as in [1], now let $p_k > 0$,k=1,2…,and x=(x_k) $\in \Delta l_\infty(p)$ implies that $\sup_k |x_k - x_{k+1}|^{p_k} < \infty$.

Now choose N such that $N > \max(1, \sup_k k^{-1}|x_k|^{p_k})$ (8)

then, we have

$$\begin{aligned}|(Ax)_n| &= \left|\sum_{k=1}^{\infty} a_{nk} x_k\right| \\ &\le \max\{ |a_{n1}||x_1|, ...|a_{nn}||x_n|, ...\} \\ &\le \sup_k\{ |a_{nk}||x_k|\} \\ &\le \sup_k k|a_{nk}|N^{1/p_k} \quad ,\text{using}(8) \\ &< \infty\end{aligned}$$

Hence $Ax \in (l_\infty)$, i.e., A $\in (\Delta l_\infty(p), l_\infty)$.

Conversely suppose that for an integer N>1

$$\sup_k k|a_{nk}|N^{1/p_k} = \infty$$

That is., $\left(k a_{nk} N^{1/p_k}\right) \notin (l_\infty(\Delta), l_\infty)$.Therefore there exists $x \in l_\infty(\Delta)$ with $\sup_k |x_k - x_{k+1}| = 1$ such that

$$\sup_k k|a_{nk}||x_k - x_{k+1}|N^{1/p_k} = \infty .$$

Hence $y = \left(N^{1/p_k} \Delta x_k\right) \in l_\infty(p)$ implies that the sequence $((Ay)_n) \notin l_\infty$,Which is a contradiction to the fact that $A \in (\Delta l_\infty(p), l_\infty)$.

Which completes the proof.

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